Elementary maths for GMT

Calculus

Part 4: Complex numbers

Complex numbers

- Equations like $x^2 = -1$ do not have a solution within the real numbers
- There are many applications where it would be useful if there *was* a solution (or at least something to do math with...)
- This requires extending the 'normal' set of real numbers to the so-called *complex numbers*
- Complex numbers are useful in many areas
  - Including differential equations, algebra, many physics topics, electricity, gas and liquid flows
  - Many of these are of importance in modeling virtual worlds
  - Quaternions (further extensions of complex numbers) are useful in efficient 3D modeling of rotations of objects

Complex numbers

- Allow you to work with roots of negative numbers
- Give a relationship between the math of exponents and trigonometric math
  - colloquially: link e and π

Complex numbers

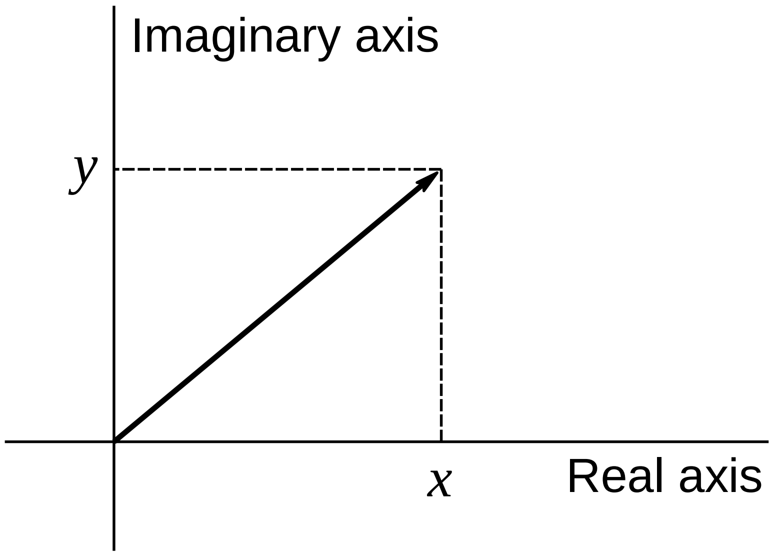
- Are created by introducing one new abstract number: $\sqrt{-1}$
- A complex number is represented by two real numbers, *x* and *y*, and can be notated as the coordinate pair (x, y) or the expression $x + iy$ where i is defined by $i^2 = -1$

Graphical representation

$$z = (x, y) = x + iy$$

Real part
$\mathrm{Re}(z) = x$

Imaginary part
$\mathrm{Im}(z) = y$



- All complex numbers (x, y) together span the *complex plane*
- Real numbers *(y=0)* are on the *real axis*
- Purely imaginary numbers *(x=0)* are on the *imaginary axis*

Complex calculus

- Rules are similar to the calculus on real numbers. Just remember that $i^2 = -1$

- Example
  - Let assume $z = x + iy$ and $w = u + iv$:
  - Addition
    $$z + w = (x + iy) + (u + iv) = x + u + iy + iv$$
    $$= (x + u) + i(y + v)$$
  - Multiplication
    $$zw = (x + iy)(u + iv) = xu + xiv + iyu + i^2yv$$
    $$= (xu - yv) + i(xv + yu)$$

Modulus and conjugate

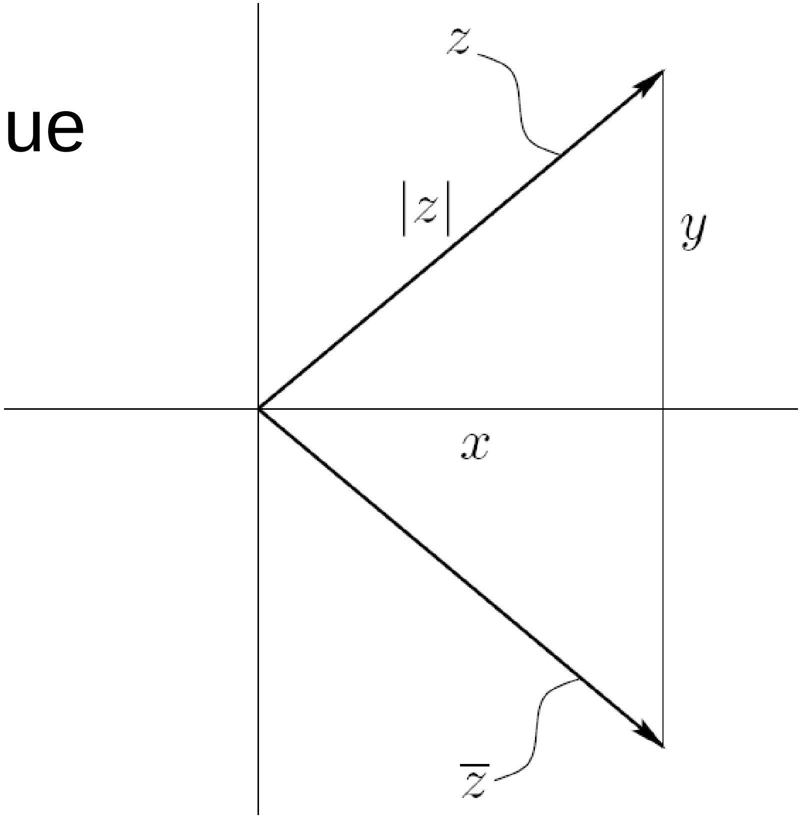
- $z = (x, y) = x + iy$

- The *modulus* or absolute value is defined by

$$|z| = \sqrt{x^2 + y^2}$$

- The *conjugate* is defined by

$$\bar{z} = x - iy$$



Conjugate

- The conjugate is important to ‘switch’ between complex and real numbers. Multiplying a complex number by its conjugate gives a real number (no i)

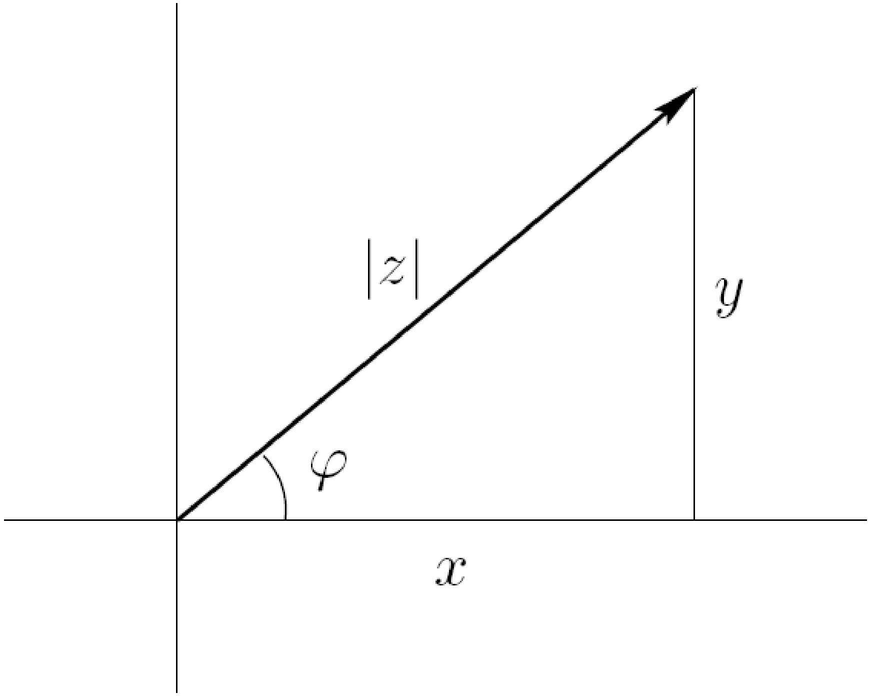
$$z\bar{z} = (x + iy)(x - iy) = x^2 - xiy + iyx - i^2y^2 = x^2 + y^2$$

- Example
  - convert a complex fraction to a $x + iy$ form

$$\frac{1+2i}{1+i} = \frac{1+2i}{1+i} \cdot \frac{1-i}{1-i} = \frac{1-i+2i-2i^2}{1^2+1^2} = \frac{3+i}{2} = \frac{3}{2} + \frac{1}{2}i$$

Polar form

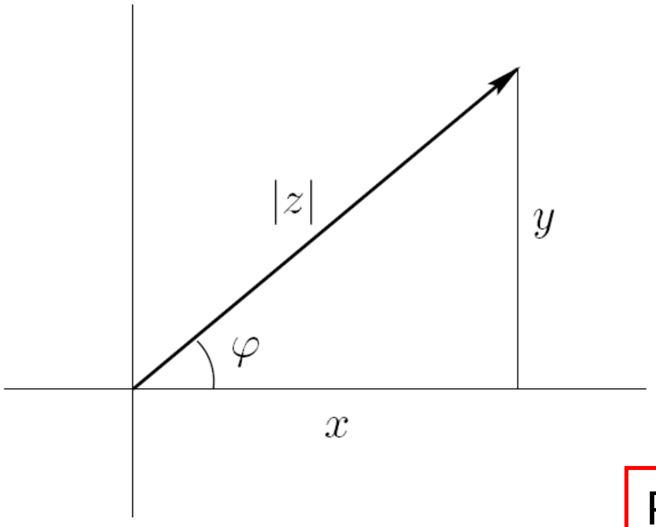
- A complex number $z = x + iy$ can also be expressed using its modulus $|z|$ and the angle φ with the positive x-axis



Polar form

- Because

$$\begin{cases} x = |z|\cos(\varphi) \\ y = |z|\sin(\varphi) \end{cases}$$



we can write

$$z = x + iy = |z|(\cos(\varphi) + i\,sin(\varphi))$$

Polar form

If φ is specified so that $-\pi < \varphi \leq \pi$, then φ is called the *argument* of *z*, denoted by *arg(z)*

Example

- Write $z = 1 + i$ in polar form
  - $x = 1$ and $y = 1$
  - $|z| = \sqrt{1^2 + 1^2} = \sqrt{2}$
  - the argument can be computed from (e.g.):

$$\cos(\varphi) = \frac{x}{|z|} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \quad \text{so} \quad \varphi = \frac{\pi}{4}$$

  - the polar form is

$$|z|(\cos(\varphi) + i\sin(\varphi)) = \sqrt{2}(\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right))$$

Euler's formula

- $\cos(\varphi) + i\sin(\varphi) = e^{i\varphi}$

  (without proof)

- So the polar form can be abbreviated to:

$$z = |z|\,(\cos(\varphi) + i\sin(\varphi))$$
$$= |z|\,e^{i\varphi}$$

Exponential polar form

Examples

- $z = |z|(\cos(\varphi) + i\sin(\varphi)) = |z|\, e^{i\varphi}$

- $3 + i\sqrt{3} = 2\sqrt{3}\left(\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)\right) = 2\sqrt{3}\, e^{i\frac{\pi}{6}}$
- $-1 = -1 + 0 \cdot i = 1(\cos(\pi) + i\sin(\pi)) = e^{i\pi}$

$$-1 = e^{i\pi}$$

A relation between $e, 1, i$ and π !!! How cool is that!

De Moivre's formula

- By multiplying complex numbers in their polar form we find

  $$|z_1 z_2| = |z_1||z_2|$$

  $$\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$$

- This generalizes to, with $z = |z|\,(\cos(\varphi) + i\sin(\varphi))$

  $$z^n = |z|^n\,(\cos(n\,\varphi) + i\sin(n\,\varphi))$$

- For unit modulus, this gives *de Moivre's formula*

  $$(\cos(\varphi) + i\sin(\varphi))^n = \cos(n\,\varphi) + i\sin(n\,\varphi)$$

Example

- For *n = 2*

$$(\cos(\varphi) + i\sin(\varphi))^2 = \cos(2\varphi) + i\sin(2\varphi)$$

$$(\cos(\varphi) + i\sin(\varphi))^2 =$$
$$cos^2(\varphi) + 2i\cos(\varphi)\sin(\varphi) + i^2 sin^2(\varphi) =$$
$$cos^2(\varphi) - sin^2(\varphi) + 2i\cos(\varphi)\sin(\varphi)$$

Taking the real and imaginary parts

$$\begin{cases} 2\cos(\varphi)\sin(\varphi) = \sin(2\varphi) \\ cos^2(\varphi) - sin^2(\varphi) = \cos(2\varphi) \end{cases}$$

➢ Very real results using complex algebra